e-MAIL ORDER

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The daffy secretary

A secretary has n letters and n addressed envelopes. Instead of matching each letter with the corresponding envelope, she inserts the letters in a random manner. What are the chances that every letter will be in the wrong envelope?

To solve a problem like this, it is often a good idea to look first at a simple case. So, let us suppose that $n = 3$. Then letters A, B, C can be placed in envelopes 1, 2, 3 in the following ways.

Envelope 1	A	A	B	B	C	C
Envelope 2	B	C	A	C	A	B
Envelope 3	C	B	C	A	B	A

We see that there are 6 = 3! possible ways of placing the letters. If we assume that the correct matching is 1 – A, 2 – B, 3 – C, then there are just two ways for which every letter is in the wrong envelope (which two?). Thus the answer to our question is two chances in six, or $\frac{2}{6}$.

1. Repeat the above working for $n = 4$. You might also try to obtain a result for $n = 5$, but preferably not by constructing an immense table! What shortcuts can you devise?

Let $W(n)$ denote the probability (chance) of putting n letters in n wrong envelopes. You should have obtained for $n = 4$, $W(4) = \frac{9}{24}$ ways, and for $n = 5$, $W(5) = \frac{44}{120}$ ways. We might also observe that $W(2) = \frac{1}{2}$ and $W(1) = 0$. There is in fact a pretty pattern here, although you could be excused for not noticing it! We note first that 6 = 3.2.1 = 3!, 24 = 4!, and 120 = 5! Now let's see what happens when we take the difference of each successive pair of results:

$$\begin{aligned}
W(1)-W(2) &= -\frac{1}{2} & & & &= \frac{-1}{2!}\\
W(2)-W(3) &= \frac{1}{2}-\frac{2}{6} & &= \frac{1}{6} & &= \frac{1}{3!}\\
W(3)-W(4) &= \frac{2}{6}-\frac{9}{24} & &= \frac{-1}{24} & &= \frac{-1}{4!}\\
W(4)-W(5) &= \frac{9}{24}-\frac{44}{120} & &= \frac{1}{120} & &= \frac{1}{5!}
\end{aligned}$$

Then our calculations suggest that:

$$\begin{aligned}
W(1) &= 0\\
&= 1-\frac{1}{1!}\\
W(2) &= W(1)+\frac{1}{2} = 0+\frac{1}{2!}\\
&= 1-\frac{1}{1!}+\frac{1}{2!}\\
W(3) &= W(2)-\frac{1}{6} = \frac{1}{2!}-\frac{1}{3!}\\
&= 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\\
W(4) &= W(3)+\frac{1}{24} = \frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\\
&= 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\\
W(5) &= W(4)-\frac{1}{120} = \frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\\
&= 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}
\end{aligned}$$

And in general,

$$W(n)=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots\pm\frac{1}{n!}$$

the sign of the last term being "+" when n is even, and "–" when n is odd.

We shall find that the two series

$$1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots$$

and

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots$$

are related in a rather unexpected way.

2. (a) In the table below we sum the first eight terms of each of the above series. Complete the table, expressing each term to four decimal places.

Term		Sum	Term		Sum
	1	1.0000		1	1.0000
Add	$\frac{1}{1!}$	2.0000	Subtract	$\frac{1}{1!}$	0.0000
Add	$\frac{1}{2!}$	2.5000	Add	$\frac{1}{2!}$	0.5000
Add	$\frac{1}{3!}$		Subtract	$\frac{1}{3!}$	
Add	$\frac{1}{4!}$		Add	$\frac{1}{4!}$	
Add	$\frac{1}{5!}$		Subtract	$\frac{1}{5!}$	
Add	$\frac{1}{6!}$		Add	$\frac{1}{6!}$	
Add	$\frac{1}{7!}$		Subtract	$\frac{1}{7!}$	

(b) Can you spot a simple (approximate) relation between your two final answers in (a)?

We write

$$e=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots$$

To 12 decimal places, e = 2.718 281 828 459.

From our calculations, it also appears to be true that

$$\frac{1}{e}=1-\frac{1}{1}+\frac{1}{1.2}-\frac{1}{1.2.3}+\ldots$$

Extensions

3. The product 1.2.3 … n is often written $n!$ (n factorial) as we have written it above. Evaluate the numbers:
$2! + 1$
$3! + 1$
$4! + 1$
$5! + 1$
$6! + 1$
Show that $n! + 1$ is not exactly divisible by any of the numbers 2, 3, 4, …, n. Is it a prime number? Always? (The idea was used by Euclid to show that there is an infinite number of prime numbers.)

4. The array of numbers below is called Pascal's triangle. Each row is obtained from the row above by a simple rule. Can you see what it is? What would the next row in the triangle be?

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \ldots & & & & &
\end{array}$$

Show that the numbers in the fifth row are given by

$$\frac{4!}{4!},\ \frac{4!}{3!1!},\ \frac{4!}{2!2!},\ \frac{4!}{1!3!},\ \frac{4!}{4!}$$

Can you express the other numbers in terms of factorials?

5. See if you can construct a mnemonic to help you remember the first few digits of the number e.

Some other patterns for e

The number e is widely occurring in mathematics. It is of interest to find and create patterns from different branches of the subject, because these increase our understanding.

Thus while it may be useful to know that

$$e = 2.718\ 281\ 828\ 459\ 045\ 235\ldots$$

this expression tells us nothing about the structure of e — how it occurs and why it is interesting. On the other hand, we find e occurring naturally in some delightfully simple sequences, series and continued fractions. Some of these we have met before.

Sequences

(a) $\left(\frac{1}{2}\right)^2, \left(\frac{2}{3}\right)^3, \left(\frac{3}{4}\right)^4, \left(\frac{4}{5}\right)^5, \ldots \to e^{-1} \left(=\frac{1}{e}\right)$

(b) $\left(\frac{2}{2}\right)^2, \left(\frac{3}{3}\right)^3, \left(\frac{4}{4}\right)^4, \left(\frac{5}{5}\right)^5, \ldots \to e^{0} \ (=1)$

(c) $\left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, \left(\frac{6}{5}\right)^5, \ldots \to e^{1} \ (=e)$

(d) $\left(\frac{4}{2}\right)^2, \left(\frac{5}{3}\right)^3, \left(\frac{6}{4}\right)^4, \left(\frac{7}{5}\right)^5, \ldots \to e^{2}$

Series

(a) $e^{-1} = 1 + \frac{(-1)}{1} + \frac{(-1)^2}{1.2} + \frac{(-1)^3}{1.2.3} + \frac{(-1)^4}{1.2.3.4} + \ldots$

(b) $e^{0} = 1 + \frac{0}{1} + \frac{0^2}{1.2} + \frac{0^3}{1.2.3} + \frac{0^4}{1.2.3.4} + \ldots$

(c) $e^{1} = 1 + \frac{1}{1} + \frac{1^2}{1.2} + \frac{1^3}{1.2.3} + \frac{1^4}{1.2.3.4} + \ldots$

(d) $e^{2} = 1 + \frac{2}{1} + \frac{2^2}{1.2} + \frac{2^3}{1.2.3} + \frac{2^4}{1.2.3.4} + \ldots$

(e) $e^{x} = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots$

(x any real number)

Continued fractions

(a) $$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

(b) $$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \ldots}}}}}$$

(c) $$\sqrt{e} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

We shall investigate this very special number e further in coming issues of AMT.

Bibliography

Scott, P. R. (1974). Discovering the mysterious numbers. Cheshire.

The exponential series : http://www.ucl.ac.uk/Mathematics/geomath/level2/series/ser9.html

From Helen Prochazka's Scrapbook

The Pythagoreans, who were the first to take up mathematics, not only advanced this subject but, saturated with it, they fancied that the principles of mathematics were the principles of all things.

Aristotle, Greek philosopher (384–322 BC)